Unexpected features of quantum degeneracies in a pairing model with two integrable limits

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The evolution pattern of level crossings and exceptional points is studied in a non-integrable pairing model with two different integrable limits. One of the integrable limits has two independent parameter-dependent integrals of motion. We demonstrate, and illustrate in our model, that quantum integrability of a system with more than one parameter-dependent integral of motion is always signaled by level crossings of a complex-extended Hamiltonian. We also find that integrability implies a reduced number of exceptional points. Both properties could uniquely characterize quantum integrability in small Hilbert spaces.

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The search for fingerprints of the chaotic/regular dynamics in the quantum regime is often focused on studies of spectral properties of the quantum systems. In this context, spectral fluctuations were intensely studied for various quantum systems. These studies lead to the BGS conjecture [1] that in the semiclassical limit the spectral fluctuations of chaotic systems are described by random matrix theory. For quantum integrable systems, Berry and Tabor [2] showed that the spectral fluctuations are well described by a Poisson statistic. While chaotic systems are characterized by level repulsion between successive levels, levels of integrable systems are uncorrelated allowing crossings between states of the same symmetries.

Level crossings and degeneracies are important for the understanding of spectral fluctuations [3] and the onset of quantum chaos [4]. Much effort has been devoted to studies of degeneracies associated with avoided crossings in quantal spectra, focusing mainly on the topological structure of the Hilbert space and the geometric phases [5, 6]. Among these degeneracies, one finds a diabolic point where two Riemann sheets of eigenvalues touch each other [5, 7], and an exceptional point (EP) [8, 9, 10] where the two sheets are entangled by the square-root type of singularity. EP appears in the complex g plane of a generic Hamiltonian $H(g) = H_0 + gH_1$, where both H_0 and H_1 are hermitian and $[H_0, H_1] \neq 0$. In many-body systems, EPs have been studied in schematic models like the Lipkin model [11] and the interacting boson model along the line connecting the dynamical symmetries U(5) to O(6) [12]. Both models can be considered as particular examples of 2-level boson pairing models pertaining to the general class of integrable Richardson-Gaudin (RG) models [13, 14]. In spite of the fact that level crossings of eigenstates with the same global symmetries are only allowed for integrable systems, 2-level pairing models have no level crossing and all degeneracies take place in the complex plane as EPs.

In this work, we will introduce a prototypical quantum integrable system, the 3-level RG model, to discuss the appearance of level crossings and EPs, and their evolution both with complex parameters of the Hamiltonian and with a non-integrable complex perturbation. On this basis, we prove that integrable models with at least two parameter-dependent integrals of motion (IMs) have a level crossing in the complex-extended parameter space, providing a clear signal of their integrability. Furthermore, the inclusion of a non-integrable perturbation splits each level crossing into two EPs transforming dramatically the topology of the Hilbert space close to the level crossing.

Let us begin by briefly reviewing the RG models which are based on the $SU(2)$ algebra with elements K_l^+, K_l^-, and K_l^0, fulfilling the commutation relations: $[K_l^+, K_{l'}^-] = \delta_{ll'} K_l^0$, $[K_l^0, K_{l'}^\pm] = \pm\delta_{ll'} K_l^\pm$. The indices l, l' refer to a particular copy from a set of L, $SU(2)$ algebras. Each $SU(2)$ algebra possesses one quantum degree of freedom. Therefore, a quantum integrable model requires the existence of L independent, global operators that commute with each other. These operators, which need not be hermitian, are the IMs. In the following, we will work with the rational family of RG models whose IMs are [13]:

$$R_l = K_l^0 + 4g \sum_{l'(\neq l)} \frac{1}{\varepsilon_l - \varepsilon_{l'}} \left[\frac{1}{2} \left(K_l^+ K_{l'}^- + K_l^- K_{l'}^+ \right) + K_l^0 K_{l'}^0 \right], \quad (1)$$

where g and ε_l are $L+1$ arbitraty parameters. The IMs (1) satisfy $[R_i, R_j] = 0$ for all pairs i, j.

There is a profound relation between quantum integrability in finite systems and the existence of level crossings. In the context of the 6-sites Hubbard model, this problem has been addressed by Yuzbashyan *et al.* [15] who showed that a level crossing implies the existence of two independent parameter-dependent IMs. Here we complete this analysis by showing that a system with at least two parameter-dependent IMs has of necessity a level crossings in the complex plane.

Let us assume a Hamiltonian $H(g)$ depending linearly on a parameter g. $H(g)$ itself is the parameter-dependent IM. If g is complex then the Hamiltonian and eventually other parameter-dependent IMs will be non-hermitian. The parameter-dependent IM $Q(g)$ commuting with the Hamiltonian: $[H(g), Q(g)] = 0$, will be independent of the Hamiltonian if it cannot be expressed as an entire function of $H(g)$ and g, i.e., $Q(g) \neq f(g, H(g))$. Let us assume that n is the dimension of the Hilbert space and $E_1(g), \cdots, E_n(g)$ and $q_1(g), \cdots, q_n(g)$ are the corresponding eigenvalues in the basis in which both operators are diagonal. If $Q(g)$ is an entire function of $H(g)$, then it can be expanded for any complex g value as:

$$\begin{aligned} q_1(g) &= a_n E_1^{n-1}(g) + \cdots + a_2 E_1(g) + a_1 \\ &\vdots \\ q_n(g) &= a_n E_n^{n-1}(g) + \cdots + a_2 E_n(g) + a_1. \end{aligned} \quad (2)$$

This set of equations has always a solution unless for some value $g = g_0$ a pair of equations $\{k, k'\}$ have $E_k(g_0) = E_{k'}(g_0)$, but $q_k(g_0) \neq q_{k'}(g_0)$, implying a double degeneracy in the Hamiltonian but not in the second IM.

The minimal rational RG model (1) allowing for level crossings should have at least three SU(2) copies. The reason is that the sum of the IMs (1) is a global conserved symmetry: $K^0 = \sum_{i=1}^{L} K_i^0$, commuting with all IMs and independent of the parameter g. Hence, we are left with two parameter-dependent IMs and, as shown above, at least two parameter-dependent IMs are required for having level crossings.

In what follows, we will use the pair representation of the SU(2) algebra leading to pairing Hamiltonians. The elementary operators in this representation are the number operators $N_j = \sum_m a_{jm}^\dagger a_{jm}$ and the pair operators $A_j^\dagger = \sum_m a_{jm}^\dagger a_{j\overline{m}}^\dagger$, where j is the total angular momentum and m is the z-projection. The state $j\overline{m}$ is the time reversal of jm.

The relation between the operators of the pair algebra and the generators of the SU(2) algebra is: $K_j^0 = \frac{1}{2}N_j - \frac{1}{4}\Omega_j$, $K_j^+ = \left(K_j^-\right)^\dagger = \frac{1}{2}A_j^\dagger$, where Ω_j is the particle degeneracy of level j. With this correspondence, one can introduce an integrable 3-level pairing Hamiltonian as:

$$H(g) = 2\sum_i \varepsilon_i R_i(g) + C \equiv \sum_i \varepsilon_i N_i + g \sum_{ij} A_i^\dagger A_j, \quad (3)$$

where C is an irrelevant constant and ε_i $(i = 1, 2, 3)$ are the single-particle energies. One can see that $H(g)$ itself is a parameter-dependent IM. As discussed above, the sum of the IMs (1) yields a parameter-independent IM, the particle number: $N = 2\sum_l R_l(g) + \frac{1}{2}\sum_l \Omega_l$. The second parameter-dependent IM can be chosen as linearly independent from the two other IMs. The simplest choice is R_1. If $\varepsilon_1 = 0$, the second parameter-dependent IM becomes:

$$\begin{aligned} Q(g) &= \left[1 + g\left(\frac{\Omega_2}{\varepsilon_2} + \frac{\Omega_3}{\varepsilon_3}\right)\right]\frac{N_1}{2} + \frac{g\Omega_1}{\varepsilon_2}\frac{N_2}{2} + \frac{g\Omega_1}{\varepsilon_3}\frac{N_3}{2} \\ &- g\left\{\frac{1}{\varepsilon_2}\left[\frac{1}{2}\left(A_1^\dagger A_2 + A_2^\dagger A_1\right) + N_1 N_2\right]\right. \\ &+ \left.\frac{1}{\varepsilon_3}\left[\frac{1}{2}\left(A_1^\dagger A_3 + A_3^\dagger A_1\right) + N_1 N_3\right]\right\}. \end{aligned} \quad (4)$$

The position of all degeneracies in the complex-g plane are indicated by the roots of the coupled equations:

$$\det\left[H\left(g\right) - EI\right] = 0 \ ; \qquad \frac{\partial}{\partial E}\det\left[H\left(g\right) - EI\right] = 0. \quad (5)$$

By eliminating E from these two equations, we are left with the discriminant $D(g)$, a polynomial in g of degree $M \leq n(n-1)$. The discriminant can be written as [9]:

$$D(g) = \prod_{m<m'}\left[E_m(g) - E_{m'}(g)\right]^2 \quad , \quad (6)$$

where $E_m(g)$, $E_{m'}(g)$ denote the complex eigenvalues of $H(g)$. The eigenvalue degeneracies $E_m(g) = E_{m'}(g)$ at $g = g_\alpha$ $(\alpha = 1, \ldots, M)$, can be found numerically by looking for sharp minima of $D(g)$. It turns out that for real values of ε_j, the roots are real or complex conjugate pairs. One finds two kinds of solutions in quantum integrable models: (i) single roots corresponding to EPs that are common to all IMs, and (ii) double- (multiple-) root solutions which indicate non-singular sharp crossings of two (or more) levels with two (or more) orthogonal wave functions, related to the existence of at least two independent IMs.

Fig. 1 shows an evolution of two level crossings, the double-roots of $D(g)$, in the complex-g plane as a function of the energy of the third single-particle level ε_3 $(\varepsilon_3 > \varepsilon_2)$. The systems has 4 pairs of fermions in a valence space with level degeneracies $\Omega_1 = 6, \Omega_2 = 4, \Omega_3 = 2$ and $\varepsilon_1 = 0$, $\varepsilon_2 = 1$. In the limit $\varepsilon_3 \to \infty$, this system decouples effectively into the two 2-level models: the first one with level ε_3 occupied and the second one with ε_3 empty. In this limit level crossings are forbidden and, indeed, the two level crossings that appear for finite values of ε_3 move to $\pm\infty$. With decreasing ε_3, two level

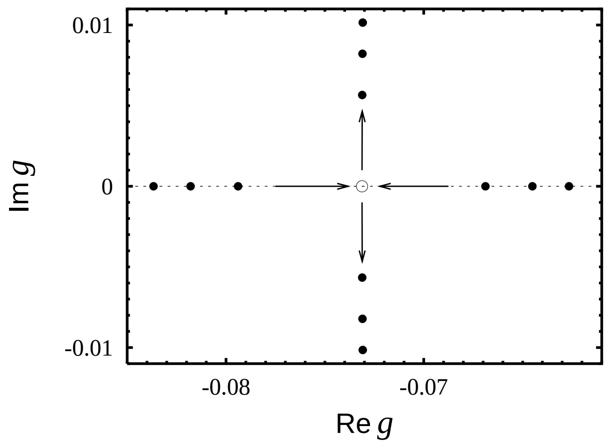

FIG. 1: Collision and subsequent scattering of two level crossings as a function of the energy ε_3 in a complex-extended integrable 3-level pairing Hamiltonian (3). Points are plotted in a descending order of ε_3. For more details, see the discussion in the text.

crossings coming from $\pm\infty$ approach each other in the real axis up to a critical value $\varepsilon_3^{(\mathrm{cr})} = 1.8499$ where they coalesce. The level crossing at this point corresponds to the quadruple-root of $D(g)$. For $\varepsilon_3 < \varepsilon_3^{(\mathrm{cr})}$ this crossing splits again into the two double-root level crossings which move into the complex-g plane. The presence of such level crossings in the complex plane is a clear signature of quantum integrability, and shows the necessity of extending the demonstration of operator independency given in (2) to the whole complex parameter space.

EPs associated to single roots of the discriminant $D(g)$, unlike level crossings, are common to all parameter-dependent IMs including the Hamiltonian. It should be also noted that EPs appear in the quantum integrable model even though no manifestation of level repulsion is expected and the spectral fluctuations of the hermitian Hamiltonian obey a Poisson distribution [16]. In that sense, level repulsion may be a sufficient but not a necessary condition for the appearance of EPs.

Another feature that we found in this 3-level pairing model, as well as in more general multi-level pairing models, is the reduction of the total number of discriminant roots whose maximum value is $M_{max} = n(n-1)$. In the particular case shown in Fig. 1, $M_{max} = 20$ but we found 16 roots consisting of 2 level crossings (double roots) and 12 EPs (single roots).

In the following, we generalize the 3-level pairing Hamiltonian (3) to study effects of non-integrability:

$$H(g) = \sum_i \varepsilon_i N_i + \zeta g \sum_{ij} A_i^\dagger A_j - (1-\zeta) g \sum_i N_i^2 \ . \quad (7)$$

For $\zeta = 1$, Eq. (7) corresponds to the integrable Hamiltonian (3). For $\zeta = 0$, the Hamiltonian (7) is also integrable with the number operators N_i playing the role of parameter-independent IMs. In the interval: $0 < \zeta < 1$, the Hamiltonian (7) is non-integrable, i.e. it does not possess an independent IM other than the Hamitonian and the total number operator. Two main features characterize the emergence of non-integrability, firstly the level crossings break into pairs of EPs and secondly the missing roots of the discriminant come into play as EPs from ∞.

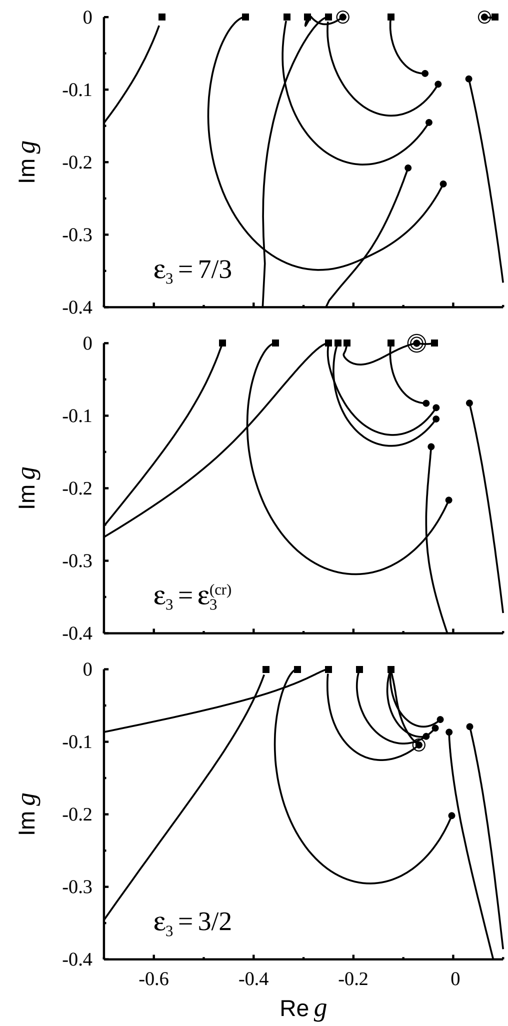

FIG. 2: The evolution of level crossings and EPs of the complex-extended 3-level pairing Hamiltonian (7) in the interval $0 \leq \zeta \leq 1$. Circles and squares denote the position of degeneracies at $\zeta = 1$ and 0, respectively. The double circles depict the double-root level crossing at $\zeta = 1$. The triple circle shows the quadruple-root level crossing corresponding to the coalescence of two double-root level crossings. For more details, see the caption of Fig. 1 and the discussion in the text.

Fig. 2 shows the global pattern of level crossings and EPs as a function of the parameter ζ ($0 \leq \zeta \leq 1$) for the 3-level system of Fig. 1 in three regimes: $\varepsilon_3 > \varepsilon_3^{(cr)}$ ($\varepsilon_3 = 7/3$), $\varepsilon_3 = \varepsilon_3^{(cr)}$, and $\varepsilon_3 < \varepsilon_3^{(cr)}$ ($\varepsilon_3 = 3/2$). Energies of levels '1' and '2' are fixed at: $\varepsilon_1 = 0, \varepsilon_2 = 1$. Only

the lower half-plane of g is shown where all eigenvalues are either discrete states on the real-g axis or else decaying resonances. Complex conjugate degeneracies situated in the upper half-plane ($\mathcal{I}m(g) > 0$) correspond to capturing resonances. In the limit of $\zeta = 1$, the two level crossings for $\varepsilon_3 = 7/3$ are located along the real-g axis. One may also notice a quadruple-root level crossing on the real-g axis at $\varepsilon_3 = \varepsilon_3^{(cr)}$ (see also Fig. 1). Moreover, one can see the location of 6 EPs of the integrable pairing Hamiltonian (3) in all regimes of ε_3.

With decreasing ζ, one observes several distinct effects. Firstly, each double-root level crossing at $\zeta = 1$ breaks into a pair of EPs. For $\varepsilon_3 = 3/2$, two EPs resulting from this fragmentation follow independent trajectories in the complex-g plane and end up in two different level crossings at $\zeta = 0$. For $\varepsilon_3 = 7/3$, the level crossing at $\zeta = 1$ breaks into two EPs symmetrically with respect to the real-g axis. Since this symmetry is conserved for all ζ, these EPs end up in the same level crossing for $\zeta = 0$. Secondly, roots that are missing in the integrable limit ($\zeta = 1$) appear from $g = \infty$ and end up in level crossings for $\zeta \to 0$. Thirdly, in the limit $\zeta \to 0$, all EPs either collapse in different level crossings at the real-g axis or escape to infinity. The double level crossings found in this limit correspond to the two different eigenvalue degeneracies. For $\varepsilon_3 = 3/2$, one can see three EPs converging on each side of the real-g axis to a single point. At this sextuple-root of the discriminant, one finds a sharp crossing of three eigenvalues.

The degree of the discriminant, which for $\zeta = 0, 1$ equals 16 in all regimes of ε_3, becomes $M = n(n-1) = 20$ in the non-integrable regime. Notice the absence of EPs in the integrable case $\zeta = 0$ reflecting the fact that the Hamiltonian is diagonal in the original basis for any g value.

In conclusion, we have shown that a system with two independent parameter-dependent IMs has at least one level crossing in the complex parameter space. We have used a minimal integrable pairing model consisting of three levels to exemplify this issue. Moreover, we have found that the degree of the discriminant is reduced in this integrable limits, giving rise to a lower number of EPs, but still a fraction of them persists in spite of the fact that there is no manifestation of level repulsion. If integrability is broken by the inclusion of a non-integrable perturbation, all level crossings split into EPs and other EPs come into the complex plane from ∞ recovering the maximal degree $M_{max} = n(n-1)$ of the discriminant. We conjecture that these two unexpected properties, level crossings in the complex-g plane and the reduction in the number of EPs, uniquely define a quatum integrable system. If this conjecture proves to be true, then it might be particularly useful in studies of finite systems with small dimensional Hilbert spaces, where the analysis of spectral fluctuations is unreliable. More work has to be done to elucidate the relation between generic integrable Hamiltonians and the missing roots of the discriminant in order to confirm this conjecture.

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